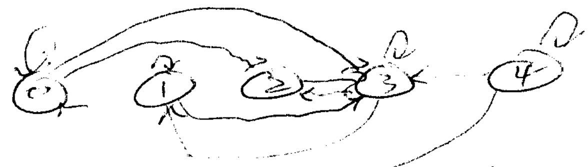
IENG 362
Stochastic Models
Markov Chains Sample Problems

1. Consider the following (one-step) transition matrix of a Markov chain.

$$\begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ P\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0.33 & 0.00 & 0.33 & 0.33 & 0.00 \\ 0.00 & 0.80 & 0.00 & 0.20 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 & 0.00 \\ 0.25 & 0.00 & 0.50 & 0.25 & 0.00 \\ 0.00 & 0.50 & 0.00 & 0.15 & 0.35 \end{bmatrix} \end{array}$$



all states communicate
{0,2,3} recurrent
{1} transient
{4} transient

Determine the classes of the Markov chain. Indicated whether they are transient or recurrent and determine the period for each class.

2. Consider the following one step transition matrix for a Markov Chain.

$$P=\begin{bmatrix} 0.75 & 0.25 & 0.00 \\ 0.25 & 0.50 & 0.25 \\ 0.30 & 0.00 & 0.70 \end{bmatrix}$$

$\pi_0 = .75\pi_0 + .25\pi_1 + .3\pi_2$
$\pi_1 = .25\pi_0 + .5\pi_1$
$\pi_2 = .25\pi_1 + .7\pi_2$
let $\pi_1 = 1$
$\Rightarrow \pi_2 = .83,\ \pi_0 = 2$

$\pi_0 = 2$, $\pi_1 = 1$, $\pi_2 = .83$, $\overline{3.83}$ $\Rightarrow$ $\pi_0 = .522$, $\pi_1 = .261$, $\pi_2 = .218$

normalize by dividing by 3.83

Find the steady state probabilities for this model.

alt. $P^{16} \Rightarrow \begin{pmatrix} .522 & .261 & .217 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$

3. Find the expected first passage time from μ_{20}, μ_{21}, and μ_{22} for the Markov Chain with states 0, 1, and 2 and one step transition probabilities given below..

$$\begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ P=\ 1 \\ 2 \end{array} & \begin{bmatrix} 0.0 & 0.3 & 0.7 \\ 0.9 & 0.1 & 0.0 \\ 0.2 & 0.0 & 0.8 \end{bmatrix} \end{array}$$

$P^{16} = \begin{vmatrix} .207 & .069 & .724 \\ '' & '' & '' \\ '' & '' & '' \end{vmatrix} \Rightarrow \pi_2 = .725$
$\Rightarrow \mu_{22} = \frac{1}{\pi_2} = 1.38$

$\mu_{ij} = 1 + \sum_{k \neq j} p_{ik}\mu_{kj}$

$\mu_{20} = 1 + p_{21}\mu_{10} + p_{22}\mu_{20}$
$= 1 + .8\mu_{20}$
$\Rightarrow \mu_{20} = \frac{1}{.2} = 5$

$\mu_{21} = 1 + p_{20}\mu_{01} + p_{22}\mu_{20}$
$= 1 + .2\mu_{01} + .8\mu_{20}$
$\mu_{21} = 4 + .2\mu_{01}$
$= 4 + .2(4.5) = 4.9$

Now
$\mu_{01} = 1 + p_{00}\mu_{01} + p_{02}\mu_{20}$
$= 1 + .7\mu_{20}$
$= 1 + .7(5) = 4.5$

$\mu_{20} = 5$
$\mu_{22} = 1.38$
$\mu_{21} = 4.9$

4. The IE in charge of a robotic assembly station believes the productivity at the station can be modeled as a Markov chain where State 0, 1, 2, or 3 indicate the number of operational robots working at the station. The time frame is one production day at the facility. A work-sampling plan provides the one-step transition probabilities.

$$P = \begin{vmatrix} 0.2 & 0.8 & 0.0 & 0.0 \\ 0.1 & 0.8 & 0.1 & 0.0 \\ 0.0 & 0.2 & 0.7 & 0.1 \\ 0.0 & 0.0 & 0.3 & 0.7 \end{vmatrix}$$

$$E[P\text{rod}] = \sum_{j=1}^{m} \pi_j c_j$$

$$= .07(0) + .56(200) + .28(400) + .09(600)$$

You are also given that

$$= 278$$

$$P^{20} = \begin{vmatrix} 0.07 & 0.56 & 0.28 & 0.09 \\ 0.07 & 0.56 & 0.28 & 0.09 \\ 0.07 & 0.56 & 0.28 & 0.09 \\ 0.07 & 0.56 & 0.28 & 0.09 \end{vmatrix}$$

Assuming the Markov model, one-step transition probabilities are correct, and the 20 step transition probabilities are correct, find the expected productivity for the station if each operational robot can assemble 200 units during the production day. Assume that a robot that fails during the day fails at the end of the day and assembles 200 units.

$$\begin{vmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0.3 & 0 & 0.7 \end{vmatrix} \text{x} \begin{vmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0.3 & 0 & 0.7 \end{vmatrix} = \begin{vmatrix} 0.625 & 0.313 & 0.063 \\ 0.388 & 0.313 & 0.300 \\ 0.435 & 0.075 & 0.490 \end{vmatrix}$$

$$\begin{vmatrix} 0.625 & 0.313 & 0.063 \\ 0.388 & 0.313 & 0.300 \\ 0.435 & 0.075 & 0.490 \end{vmatrix} \text{x} \begin{vmatrix} 0.625 & 0.313 & 0.063 \\ 0.388 & 0.313 & 0.300 \\ 0.435 & 0.075 & 0.490 \end{vmatrix} = \begin{vmatrix} 0.539 & 0.298 & 0.163 \\ 0.494 & 0.241 & 0.265 \\ 0.514 & 0.196 & 0.290 \end{vmatrix}$$

$$\begin{vmatrix} 0.539 & 0.298 & 0.163 \\ 0.494 & 0.241 & 0.265 \\ 0.514 & 0.196 & 0.290 \end{vmatrix} \text{x} \begin{vmatrix} 0.539 & 0.298 & 0.163 \\ 0.494 & 0.241 & 0.265 \\ 0.514 & 0.196 & 0.290 \end{vmatrix} = \begin{vmatrix} 0.521 & 0.264 & 0.214 \\ 0.521 & 0.257 & 0.221 \\ 0.523 & 0.257 & 0.220 \end{vmatrix}$$

$$\begin{vmatrix} 0.521 & 0.264 & 0.214 \\ 0.521 & 0.257 & 0.221 \\ 0.523 & 0.257 & 0.220 \end{vmatrix} \text{x} \begin{vmatrix} 0.521 & 0.264 & 0.214 \\ 0.521 & 0.257 & 0.221 \\ 0.523 & 0.257 & 0.220 \end{vmatrix} = \begin{vmatrix} 0.522 & 0.261 & 0.217 \\ 0.522 & 0.261 & 0.217 \\ 0.522 & 0.261 & 0.217 \end{vmatrix}$$

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